

# Gaussian lower bounds for non-homogeneous Kolmogorov equations with measurable coefficients

Alberto Lanconelli\*

Andrea Pascucci<sup>†</sup>Sergio Polidoro<sup>‡</sup>

This version: April 25, 2017

## Abstract

We prove Gaussian upper and lower bounds for the fundamental solutions of a class of degenerate parabolic equations satisfying a weak Hörmander condition. The bound is independent of the smoothness of the coefficients and generalizes classical results for uniformly parabolic equations.

**Keywords:** Kolmogorov equations, fundamental solution, linear stochastic equations, Harnack inequalities.

## 1 Introduction

We consider the Kolmogorov backward equation

$$\mathcal{L}u := \sum_{i,j=1}^{m_0} \partial_{x_i} (a_{ij} \partial_{x_j} u) + \sum_{i=1}^{m_0} (\partial_{x_i} (a_i u) + b_i \partial_{x_i} u) + cu + \sum_{i,j=1}^d b_{ij} x_j \partial_{x_i} u + \partial_t u = 0, \quad (1.1)$$

where  $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ ,  $m_0 \leq d$  and  $\mathcal{L}$  verifies the following two standing assumptions:

**Assumption 1.1.** *The coefficients  $a_{ij} = a_{ji}$ ,  $a_i, b_i, c$ , for  $1 \leq i, j \leq m_0$ , are bounded, measurable functions of  $(t, x) \in \mathbb{R} \times \mathbb{R}^d$  and*

$$\mu^{-1} |\xi|^2 \leq \sum_{i,j=1}^{m_0} a_{ij}(t, x) \xi_i \xi_j \leq \mu |\xi|^2, \quad \xi \in \mathbb{R}^{m_0}, \quad (t, x) \in \mathbb{R}^{d+1}, \quad (1.2)$$

for some positive constant  $\mu$ .

---

\*Dipartimento di Matematica, Università di Bari Aldo Moro, Bari, Italy. **e-mail:** alberto.lanconelli@uniba.it

<sup>†</sup>Dipartimento di Matematica, Università di Bologna, Bologna, Italy. **e-mail:** andrea.pascucci@unibo.it

<sup>‡</sup>Dipartimento di Scienze Fisiche, Informatiche e Matematiche, Università di Modena e Reggio Emilia, Modena, Italy. **e-mail:** sergio.polidoro@unimore.it

**Assumption 1.2.** *The matrix  $B := (b_{ij})_{1 \leq i,j \leq d}$  has constant real entries and takes the block-form*

$$B = \begin{pmatrix} * & * & \cdots & * & * \\ B_1 & * & \cdots & * & * \\ 0 & B_2 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_\nu & * \end{pmatrix} \quad (1.3)$$

where each  $B_i$  is a  $(m_i \times m_{i-1})$ -matrix of rank  $m_i$  with

$$m_0 \geq m_1 \geq \cdots \geq m_\nu \geq 1, \quad \sum_{i=0}^{\nu} m_i = d,$$

and the blocks denoted by “\*” are arbitrary.

Our main result extends the bounds proved in [3] and [29, 30] for uniformly parabolic operators with measurable coefficients: we refer to [15] for a description of the development of this theory for non-degenerate parabolic operators, which includes the fundamental contributions in [31] and [8].

**Theorem 1.3.** *Let  $\mathcal{L}$  be an operator in the form (1.1), satisfying Assumptions 1.1 and 1.2. Let  $I = ]T_0, T_1[$  be a bounded interval. Then, there exist four positive constants  $\lambda^+, \lambda^-, C^+, C^-$  such that*

$$C^- \Gamma^{\lambda^-}(t, x; T, y) \leq \Gamma(t, x; T, y) \leq C^+ \Gamma^{\lambda^+}(t, x; T, y) \quad (1.4)$$

for every  $(t, x), (T, y) \in \mathbb{R}^{d+1}$  with  $T_0 < t < T < T_1$ . The constants  $\lambda^+, \lambda^-, C^+, C^-$  depend only on  $d, \mathcal{L}$  and  $T_1 - T_0$ . In (1.4)  $\Gamma^{\lambda^-}$  and  $\Gamma^{\lambda^+}$  denote the fundamental solutions of  $\mathcal{L}^{\lambda^-}$  and  $\mathcal{L}^{\lambda^+}$ , respectively, where

$$\mathcal{L}^\lambda := \frac{\lambda}{2} \sum_{i=1}^{m_0} \partial_{x_i x_i} + \sum_{i,j=1}^d b_{ij} x_j \partial_{x_i} + \partial_t. \quad (1.5)$$

The explicit expression of  $\Gamma^{\lambda^\pm}$  is given in (2.8) below.

Degenerate equations of the form (1.1) naturally arise in the theory of stochastic processes, in physics and in mathematical finance. For instance, if  $W$  denotes a real Brownian motion, then the simplest non-trivial Kolmogorov operator

$$\frac{1}{2} \partial_{vv} + v \partial_x + \partial_t, \quad t \geq 0, (v, x) \in \mathbb{R}^2,$$

is the infinitesimal generator of the classical Langevin's stochastic equation

$$\begin{cases} dV_t = dW_t, \\ dX_t = V_t dt, \end{cases}$$

that describes the position  $X$  and velocity  $V$  of a particle in the phase space (cf. [27]). Notice that in this case we have  $1 = m_0 < d = 2$ .

Linear Fokker-Planck equations (cf. [10] and [38]), non-linear Boltzmann-Landau equations (cf. [28] and [6]) and non-linear equations for Lagrangian stochastic models commonly used in the simulation of turbulent flows (cf. [5]) can be written in the form

$$\sum_{i,j=1}^n \partial_{v_i} (a_{ij} \partial_{v_j} f) + \sum_{j=1}^n v_j \partial_{x_j} f + \partial_t f = 0, \quad t \geq 0, v \in \mathbb{R}^n, x \in \mathbb{R}^n, \quad (1.6)$$

with the coefficients  $a_{ij} = a_{ij}(t, v, x, f)$  that may depend on the solution  $f$  through some integral expressions. Clearly (1.6) is a particular case of (1.1) with  $n = m_0 < d = 2n$  and

$$B = \begin{pmatrix} 0_n & 0_n \\ I_n & 0_n \end{pmatrix}$$

where  $I_n$  and  $0_n$  denote the  $(n \times n)$ -identity matrix and the  $(n \times n)$ -zero matrix, respectively.

In mathematical finance, equations of the form (1.1) appear in various models for the pricing of path-dependent derivatives such as Asian options (cf., for instance, [32], [4]), stochastic volatility models (cf. [19], [35]) and in the theory of stochastic utility (cf. [1], [2]).

Besides its applicative interest, the operator  $\mathcal{L}$  in (1.1) has been studied by several authors because of its challenging theoretical features. As in the study of uniformly parabolic operators, the theoretical results mainly depend on the assumptions on the coefficients. We summarize here the main results available in the literature and we focus in particular on those that are useful for the purpose of this work:

- *Constant coefficients.* If the  $a_{ij}$ 's, the  $a_i$ 's and the  $b_i$ 's are constant and  $c = 0$ , the operator  $\mathcal{L}$  appears as the prototype of *hypoelliptic operators* in the seminal Hörmander's work [20]. In particular, Hörmander proves that a smooth fundamental solution for  $\mathcal{L}$  exists if, and only if, Assumptions 1.1 and 1.2 are satisfied. We emphasize that this regularity property is not obvious for strongly degenerate operators of the form (1.1). Based on the explicit expression of the fundamental solution, mean value formulas and Harnack inequalities for the non-negative solutions of  $\mathcal{L}u = 0$  have been proved in [23, 24, 16, 26]. In particular, [26] studies the invariance of the solutions of  $\mathcal{L}u = 0$  with respect to suitable *non-Euclidean* translations and *non-homogeneous* dilations: it is then proved a Harnack inequality which is translation- and dilation-invariant. In Section 2 we give the precise statement of the above assertions.
- *Hölder continuous coefficients.* The existence of a fundamental solution for operators  $\mathcal{L}$  with Hölder continuous coefficients has been proved by several authors using the parametrix method [41, 21, 39, 36, 11, 9]. An invariant Harnack inequality has been proved in [36, 13] and a lower bound for the fundamental solution of  $\mathcal{L}$  is obtained in [37, 13].
- *Measurable coefficients.* An upper bound for the fundamental solution of  $\mathcal{L}$  is obtained in [33, 25] by adapting the Aronson's method [3]. It is based on a local  $L^\infty$ -estimate of the

solutions based on a Moser's iterative procedure which in turn relies on the combination of a Caccioppoli inequality with a Sobolev estimate (see [34, 7, 25]). The authors of [40] prove a weak form of the Poincaré inequality which yields the  $C^\alpha$ -regularity of the solutions of  $\mathcal{L}u = 0$ . More recently, [18] and [22] independently provide an alternative proof of the  $C^\alpha$ -regularity of the weak solutions. Later, in the joint work [17], the aforementioned four authors use their result to prove an invariant Harnack inequality for the positive solutions of (1.6).

The main result of this paper is a lower bound for the fundamental solution  $\Gamma$  of  $\mathcal{L}$  by merely assuming the measurability and boundedness of its coefficients, in the spirit of the works [3] and [29, 30]. Its proof is based on the repeated application of the Harnack inequality on suitable sequences of points that are usually called *Harnack chains*. The method used in [17] seems us to be appropriate for the more general class of operators  $\mathcal{L}$  in (1.1) satisfying Assumptions 1.1 and 1.2; for this reason, we assume the validity of the invariant Harnack inequality for the positive weak solutions of  $\mathcal{L}u = 0$  and use it to prove the lower bound in (1.4) under these assumptions. This choice allows us to point out more clearly the geometric aspect of our argument. We also mention the forthcoming paper [14] which aims at extending the techniques of [17] to the more general class of the operators satisfying Assumptions 1.1 and 1.2.

## 2 Preliminaries

Hereafter the operator  $\mathcal{L}$  in (1.1) will be written in the compact form

$$\mathcal{L}u = \operatorname{div}(ADu + au) + \langle b, Du \rangle + cu + Yu = 0,$$

where  $D = (\partial_{x_1}, \dots, \partial_{x_d})$  denotes the gradient in  $\mathbb{R}^d$ ,  $A := (a_{ij})_{1 \leq i, j \leq d}$ ,  $a := (a_i)_{1 \leq i \leq d}$ ,  $b := (b_i)_{1 \leq i \leq d}$  with  $a_{ij} = a_i = b_i \equiv 0$  for  $i > m_0$  or  $j > m_0$ , and

$$Y := \langle Bx, D \rangle + \partial_t.$$

The *constant-coefficient Kolmogorov operator*

$$\mathcal{L}^1 := \frac{1}{2} \sum_{i=1}^{m_0} \partial_{x_i x_i} + Y$$

will be referred to as the *principal part of  $\mathcal{L}$* . It will be clear in the sequel that  $\mathcal{L}^1$  plays in this setting the role played by the heat operator in the uniformly parabolic case. We focus here, in particular, on the regularity properties of  $\mathcal{L}^1$  and on its invariance with respect to a family of non-Euclidean translations and non-homogeneous dilations. It is known that Assumption 1.2 is equivalent to the hypoellipticity of  $\mathcal{L}^1$ ; in fact, Assumption 1.2 is also equivalent to the well-known Hörmander's condition, which in our setting reads:

$$\operatorname{rank} \operatorname{Lie}(\partial_{x_1}, \dots, \partial_{x_{m_0}}, Y)(t, x) = d + 1, \quad \text{for all } (t, x) \in \mathbb{R}^{d+1}, \quad (2.1)$$

where  $\text{Lie}(\partial_{x_1}, \dots, \partial_{x_{m_0}}, Y)$  denotes the Lie algebra generated by the vector fields  $\partial_{x_1}, \dots, \partial_{x_{m_0}}$  and  $Y$  (see Proposition 2.1 in [26]). Thus operator  $\mathcal{L}$  can be regarded as a perturbation of its principal part  $\mathcal{L}^1$ : roughly speaking, Assumption 1.1 ensures that the sub-elliptic structure of  $\mathcal{L}^1$  is preserved under perturbation.

Constant-coefficient Kolmogorov operators are naturally associated to *linear* stochastic differential equations: indeed,  $\mathcal{L}^1$  is the infinitesimal generator of the  $d$ -dimensional SDE

$$dX_t = BX_t dt + \sigma dW_t, \quad (2.2)$$

where  $W$  is a standard  $m_0$ -dimensional Brownian motion and  $\sigma$  is the  $(d \times m_0)$ -matrix

$$\sigma = \begin{pmatrix} I_{m_0} \\ 0 \end{pmatrix}. \quad (2.3)$$

The solution  $X$  of (2.2) is a Gaussian process with transition density

$$\Gamma^1(t, x; T, y) = \frac{1}{\sqrt{(2\pi)^d \det \mathcal{C}(T-t)}} \exp \left( -\frac{1}{2} \langle \mathcal{C}(T-t)^{-1} (y - e^{(T-t)B} x), (y - e^{(T-t)B} x) \rangle \right) \quad (2.4)$$

for  $t < T$  and  $x, y \in \mathbb{R}^d$ , where

$$\mathcal{C}(t) = \int_0^t (e^{sB} \sigma) (e^{sB} \sigma)^* ds \quad (2.5)$$

is the covariance matrix of  $X_t$ . Assumption 1.2 ensures (actually, is equivalent to the fact) that  $\mathcal{C}(t)$  is positive definite for any positive  $t$ . Moreover  $\Gamma^1$  in (2.4) is the fundamental solution of  $\mathcal{L}^1$  and the function

$$u(t, x) := E[\varphi(X_T) \mid X_t = x] = \int_{\mathbb{R}^d} \Gamma^1(t, x; T, y) \varphi(y) dy, \quad t < T, \ x \in \mathbb{R}^d,$$

solves the backward Cauchy problem

$$\begin{cases} \mathcal{L}^1 u(t, x) = 0, & t < T, \ x \in \mathbb{R}^d, \\ u(T, x) = \varphi(x) & x \in \mathbb{R}^d, \end{cases}$$

for any bounded and continuous function  $\varphi$ .

Operator  $\mathcal{L}^1$  has some remarkable invariance properties that were first studied in [26]. Denote by  $\ell_{(\tau, \xi)}$ , for  $(\tau, \xi) \in \mathbb{R}^{d+1}$ , the left-translations in  $\mathbb{R}^{d+1}$  defined as

$$\ell_{(\tau, \xi)}(t, x) := (\tau, \xi) \circ (t, x) := (t + \tau, x + e^{tB} \xi), \quad (2.6)$$

Then,  $\mathcal{L}^1$  is invariant with respect to  $\ell_\zeta$  in the sense that

$$\mathcal{L}^1(u \circ \ell_\zeta) = (\mathcal{L}^1 u) \circ \ell_\zeta, \quad \zeta \in \mathbb{R}^{d+1}.$$

Moreover, let  $\mathcal{D}(r)$  be defined as

$$\mathcal{D}(r) := \text{diag}(rI_{m_0}, r^3I_{m_1}, \dots, r^{2\nu+1}I_{m_\nu}), \quad r \geq 0, \quad (2.7)$$

where  $I_{m_i}$  denotes the  $(m_i \times m_i)$ -identity matrix. Then,  $\mathcal{L}^1$  is homogeneous with respect to the dilations in  $\mathbb{R}^{d+1}$  defined as

$$\delta_r(t, x) := (r^2 t, \mathcal{D}(r)x),$$

if and only if all the  $*$ -blocks of  $B$  in (1.3) are null ([26], Proposition 2.2). In this case, we have

$$\mathcal{L}^1(u \circ \delta_r) = r^2(\mathcal{L}^1 u) \circ \delta_r.$$

The natural number

$$Q := m_0 + 3m_1 + \cdots + (2\nu + 1)m_\nu.$$

is usually called the *homogeneous dimension* of  $\mathbb{R}^d$  with respect to  $(\mathcal{D}(r))_{r>0}$ , because the Jacobian of  $\mathcal{D}(r)$  is equal to  $r^Q$ .

In accordance with (2.4), the fundamental solution of the operator  $\mathcal{L}^\lambda$  defined in (1.5) is

$$\Gamma^\lambda(t, x; T, y) = \frac{1}{\sqrt{(2\pi\lambda)^d \det \mathcal{C}(T-t)}} \exp \left( -\frac{1}{2\lambda} \langle \mathcal{C}(T-t)^{-1}(y - e^{(T-t)B}x), (y - e^{(T-t)B}x) \rangle \right) \quad (2.8)$$

for  $t < T$  and  $x, y \in \mathbb{R}^d$ .

We end this section with the definitions of weak and fundamental solutions utilized in the sequel.

**Definition 2.1.** A weak solution of (1.1) in a domain  $\Omega$  of  $\mathbb{R}^{d+1}$  is a function  $u$  such that

$$u, \partial_{x_1} u, \dots, \partial_{x_{m_0}} u, Yu \in L_{\text{loc}}^2(\Omega)$$

and

$$\int_{\Omega} -\langle ADu, D\psi \rangle - u\langle a, D\psi \rangle + \psi\langle b, Du \rangle + u\psi + \psi Yu = 0,$$

for any  $\psi \in C_0^\infty(\Omega)$ .

We recall that the formal adjoint operator of  $\mathcal{L}$  is defined as

$$\mathcal{L}^* v := \sum_{i,j=1}^{m_0} \partial_{y_i} (a_{ij} \partial_{y_j} v) - \sum_{i=1}^{m_0} (a_i \partial_{y_i} v + \partial_{y_i} (b_i v)) + (c - \text{tr}(B)) v - \sum_{i,j=1}^d b_{ij} y_j \partial_{y_i} v - \partial_T v. \quad (2.9)$$

**Definition 2.2.** A fundamental solution for  $\mathcal{L}$  is a continuous and positive function  $\Gamma = \Gamma(t, x; T, y)$ , defined for  $t < T$  and  $x, y \in \mathbb{R}^d$ , such that:

- i)  $\Gamma(\cdot, \cdot; T, y)$  is a weak solution of  $\mathcal{L}u = 0$  in  $]-\infty, T[ \times \mathbb{R}^d$  and  $\Gamma(t, x; \cdot, \cdot)$  is a weak solution of  $\mathcal{L}^* u = 0$  in  $]t, +\infty[ \times \mathbb{R}^d$ ;

ii) for any bounded function  $\varphi \in C(\mathbb{R}^d)$  and  $x, y \in \mathbb{R}^d$ , we have

$$\lim_{\substack{(t,x) \rightarrow (T,y) \\ t < T}} u(t, x) = \varphi(y), \quad \lim_{\substack{(T,y) \rightarrow (t,x) \\ T > t}} v(T, y) = \varphi(x),$$

where

$$u(t, x) := \int_{\mathbb{R}^d} \Gamma(t, x; T, y) \varphi(y) dy, \quad v(T, y) := \int_{\mathbb{R}^d} \Gamma(t, x; T, y) \varphi(x) dx. \quad (2.10)$$

**Remark 2.3.** The functions in (2.10) are weak solutions of the following backward and forward Cauchy problems:

$$\begin{cases} \mathcal{L}u(t, x) = 0, & (t, x) \in ]-\infty, T[ \times \mathbb{R}^d, \\ u(T, x) = \varphi(x), & x \in \mathbb{R}^d, \end{cases} \quad \begin{cases} \mathcal{L}^*v(T, y) = 0, & (T, y) \in ]t, +\infty[ \times \mathbb{R}^d, \\ v(t, y) = \varphi(y) & y \in \mathbb{R}^d. \end{cases}$$

**Remark 2.4.** A fundamental solution for  $\mathcal{L}$  exists under the additional condition that the coefficients are Hölder continuous (see [36, 11, 9]).

**Remark 2.5.** Let  $u$  be a weak solution of (1.1) and  $r > 0$ . Then  $v := u \circ \delta_r$  solves  $\mathcal{L}^{(r)}v = 0$  where

$$\mathcal{L}^{(r)}v := \operatorname{div}(A^{(r)}Dv) + \operatorname{div}(a^{(r)}v) + \langle b^{(r)}, Dv \rangle + c^{(r)}v + \langle B^{(r)}x, Dv \rangle + \partial_t v, \quad (2.11)$$

with  $A^{(r)} = A \circ \delta_r$ ,  $a^{(r)} = r(a \circ \delta_r)$ ,  $b^{(r)} = r(b \circ \delta_r)$ ,  $c^{(r)} = r^2(c \circ \delta_r)$  and  $B^{(r)} = r^2 \mathcal{D}_r B \mathcal{D}_{\frac{1}{r}}$ , that is

$$B^{(r)} = \begin{pmatrix} r^2 B_{1,1} & r^4 B_{1,2} & \cdots & r^{2\nu} B_{1,\nu} & r^{2\nu+2} B_{1,\nu+1} \\ B_1 & r^2 B_{2,2} & \cdots & r^{2\nu-2} B_{2,\nu} & r^{2\nu} B_{2,\nu+1} \\ 0 & B_2 & \cdots & r^{2\nu-4} B_{3,\nu} & r^{2\nu-2} B_{3,\nu+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_\nu & r^2 B_{\nu+1,\nu+1} \end{pmatrix},$$

where  $B_{i,j}$  denotes the  $*$ -block in the  $(i, j)$ -th position in (1.3).

**Notation 2.6.** Let  $M > 0$  and  $B := (b_{ij})_{1 \leq i, j \leq d}$  a matrix that satisfies Assumption 1.2. We denote by  $\mathcal{K}_{M,B}$  the class of Kolmogorov operators of the form (2.11) with  $\varrho \in [0, 1]$  and the coefficients  $a_{ij}, a_i, b_i, c$ , for  $1 \leq i, j \leq m_0$ , that satisfy Assumption 1.1 with the non-degeneracy constant  $\mu$  in (1.2) and the norms  $\|a_i\|_\infty, \|b_i\|_\infty, \|c\|_\infty$  smaller than  $M$ .

**Remark 2.7.** Let  $\mathcal{L} \in \mathcal{K}_{M,B}$ . If  $u$  is a solution of  $\mathcal{L}u = 0$  then, for any  $\zeta \in \mathbb{R}^{d+1}$ ,  $v := u \circ \ell_\zeta$  solves  $(\mathcal{L} \circ \ell_\zeta)v = 0$  where  $(\mathcal{L} \circ \ell_\zeta)$  is the operator obtained from  $\mathcal{L}$  by  $\ell_\zeta$ -translating its coefficients. Moreover, operator  $(\mathcal{L} \circ \ell_\zeta)$  still belongs to  $\mathcal{K}_{M,B}$ .

### 3 Harnack inequalities

Let  $B$  be a matrix that satisfies Assumption 1.2. We associate to  $B$  the cylinders

$$Q_1^+ = \{(t, x) \in \mathbb{R} \times \mathbb{R}^d \mid 0 \leq t < 1, |x| < 1\},$$

and

$$Q_r^+(z_0) := z_0 \circ \delta_r(Q_1^+) = \{z \in \mathbb{R}^{d+1} \mid z = z_0 \circ \delta_r(\zeta), \zeta \in Q_1^+\},$$

for  $z_0 \in \mathbb{R}^{d+1}$  and  $r > 0$ . The following remarkable result is proved in [17] for prototype Kolmogorov equations (1.6) and in [14] for general Kolmogorov equations (1.1).

**Theorem 3.1 (Local Harnack inequality).** *Let  $\mathcal{L} \in \mathcal{K}_{M,B}$ . If  $u$  is a non-negative weak solution of (1.1) in  $Q_1^+$  then*

$$\sup_{Q_r^+(\beta,0)} u \leq C \inf_{Q_r^+(0,0)} u, \quad (3.1)$$

where the constants  $C \geq 1$  and  $\beta, r \in ]0, 1[$  depend only on  $M$  and  $B$ .

**Remark 3.2.** *The constants  $\beta, r$  in Theorem 3.1 are small so that the cylinders  $Q_r^+(0,0)$  and  $Q_r^+(\beta,0)$  are disjoint subsets of  $Q_1^+$ .*

**Remark 3.3.** *By Remark 2.7, the Harnack inequality (3.1) is valid for cylinders centered at an arbitrary point  $z_0 \in \mathbb{R}^d$  with the same constants  $C, \beta, r$ , dependent only on  $M$  and  $B$ .*

Next we prove a global version of the Harnack inequality based on a classical argument which makes use of the so-called Harnack chains. We first prove a preliminary result. For  $\beta, r, R > 0$  and  $z_0 \in \mathbb{R}^{d+1}$ , we define the cones

$$P_{\beta,r,R} = \left\{ z \in \mathbb{R}^{d+1} \mid z = \delta_\varrho(\beta, \xi), |\xi| < r, 0 < \varrho \leq R \right\},$$

and  $P_{\beta,r,R}(z_0) := z_0 \circ P_{\beta,r,R}$ . Here  $|\xi|$  denotes the Euclidean norm of the vector  $\xi \in \mathbb{R}^d$ . Theorem 3.1 combined with Remark 2.5 gives the following

**Lemma 3.4.** *Let  $z \in \mathbb{R}^{d+1}$  and  $R \in ]0, 1]$ . Let  $u$  be a continuous and non-negative weak solution of (1.1) in  $Q_R^+(z)$ . Then we have*

$$\sup_{P_{\beta,r,R}(z)} u \leq Cu(z),$$

where the constants  $C, \beta$  and  $r$  are the same as in Theorem 3.1 and depend only on  $M$  and  $B$ .

*Proof.* We preliminarily recall that [40] prove the Hölder continuity of the solutions of  $\mathcal{L}u = 0$ . In particular,  $u$  is a continuous function. So let  $u$  be a continuous and non-negative weak solution of (1.1) in  $Q_R^+(z)$  and let  $w \in P_{\beta,r,R}(z)$ . Then  $w = z \circ \delta_\varrho(\beta, \xi)$  for some  $\varrho \in ]0, R]$  and  $|\xi| < r$ . By using the notation introduced in (2.6), we obtain from Remark 2.5 that the function  $u_{z,\varrho} := u \circ \ell_z \circ \delta_\varrho$  is a continuous and non-negative weak solution in  $Q_{\frac{R}{\varrho}}^+(0,0) \supseteq Q_1^+(0,0)$  of  $\mathcal{L}^{(\varrho)}u_\varrho = 0$ , where  $\mathcal{L}^{(\varrho)}$  is the operator defined in (2.11). Since  $\mathcal{L}^{(\varrho)} \in \mathcal{K}_{M,B}$ , by the Harnack inequality (3.1) for  $\mathcal{L}^{(\varrho)}$ , we have

$$u(w) = u_{z,\varrho}(\beta, \xi) \leq \sup_{Q_r^+(\beta,0)} u_{z,\varrho} \leq C \inf_{Q_r^+(0,0)} u_{z,\varrho} \leq Cu_{z,\varrho}(0,0) = Cu(z).$$

□



**Theorem 3.5 (Global Harnack inequality).** *Let  $\mathcal{L} \in \mathcal{K}_{M,B}$ ,  $T \in \mathbb{R}$  and  $\tau \in ]0, 1]$ . If  $u$  is a non-negative weak solution of (1.1) in  $]T - \tau, T + \tau[ \times \mathbb{R}^d$ , then we have*

$$u(T, y) \leq c_0 e^{c_0 \langle \mathcal{C}^{-1}(T-t)(y - e^{(T-t)B}x), y - e^{(T-t)B}x \rangle} u(t, x), \quad t \in ]T - \tau, T[, \quad x, y \in \mathbb{R}^d,$$

where  $\mathcal{C}$  is the covariance matrix in (2.5) and  $c_0$  is a positive constant that depends only on  $M$  and  $B$ .

Before proving Theorem 3.5, we recall (see, for instance, Sect.9.5 in [32]) that the Hörmander condition (2.1) is equivalent to the fact that the pair of matrices  $(B, \sigma)$ , with  $\sigma$  as in (2.3), is controllable in the following sense: for any  $(t, x), (T, y) \in \mathbb{R}^{d+1}$  with  $t < T$ , there exists  $v \in L^2([t, T]; \mathbb{R}^{m_0})$  such that the system

$$\begin{cases} \gamma'(s) = B\gamma(s) + \sigma v(s), \\ \gamma(t) = x, \quad \gamma(T) = y, \end{cases} \quad (3.2)$$

has solution. The function  $v$  is called a *control for  $(B, \sigma)$  on  $[t, T]$* . In the proof of Theorem 3.5 we will use the following

**Lemma 3.6.** *Let  $\gamma$  be the solution of the linear problem*

$$\begin{cases} \gamma'(s) = B\gamma(s) + \sigma v(s), & s \in [t, T], \\ \gamma(t) = x, \end{cases} \quad (3.3)$$

with  $T - t \leq 1$ , initial datum  $x \in \mathbb{R}^d$  and control function  $v \in L^2([t, T]; \mathbb{R}^{m_0})$ . Then we have

$$(s, \gamma(s)) \in P_{1, \kappa \|v\|_{L^2([t, T])}, \sqrt{T-t}}(t, x), \quad s \in [t, T],$$

where  $\kappa$  is a positive constant which depends only on  $B$ .

*Proof.* The explicit solution of (3.3) is

$$\gamma(s) = e^{(s-t)B}x + \int_t^s e^{(s-\tau)B} \sigma v(\tau) d\tau, \quad s \in [t, T].$$

Thus, setting  $\varrho = \sqrt{s-t}$ , we have that  $(s, \gamma(s)) \in P_{1, r, \sqrt{T-t}}(t, x)$  if and only if

$$\int_t^{t+\varrho^2} e^{(t+\varrho^2-\tau)B} \sigma v(\tau) d\tau = \mathcal{D}(\varrho) \xi \quad \text{with} \quad |\xi| \leq r. \quad (3.4)$$

To check this, we first notice that, according to (2.7), the space  $\mathbb{R}^d$  admits a natural decomposition as a direct sum

$$\mathbb{R}^d = \bigoplus_{j=0}^{\nu} V_j, \quad \dim V_j = m_j.$$

Then, for  $x \in \mathbb{R}^d$ , with obvious notation we have  $x = x^{(0)} \oplus \dots \oplus x^{(\nu)}$  where

$$\mathcal{D}(r)x^{(j)} = r^{2j+1}x^{(j)}, \quad j = 0, \dots, \nu.$$

We also write a  $(d \times d)$ -matrix  $E$  in block form as in (1.3), that is  $E = (E^{(ij)})_{i,j=0,\dots,\nu}$  where  $E^{(ij)}$  is a block of dimension  $m_i \times m_j$ . In particular, given the definition of exponential  $E(t) := e^{tB}$  as the sum of a power series, a direct computation shows that

$$\begin{aligned} E^{(00)}(t) &= I_{m_0} + tO(t), \\ E^{(0j)}(t) &= \frac{t^j}{j!} (I_{m_j} + tO(t)) B_j \cdots B_1, \quad j = 1, \dots, \nu, \end{aligned} \tag{3.5}$$

as  $t \rightarrow 0$ , where  $I_{m_j}$  denotes the  $(m_j \times m_j)$ -identity matrix. Now,  $\sigma v \in V_0$  and therefore, by (3.5), we have

$$\left| \left( e^{(t+\varrho^2-\tau)B} \sigma v(\tau) \right)^{(j)} \right| \leq \kappa (t + \varrho^2 - \tau)^j |v(\tau)|, \quad \tau \in [t, T],$$

with the constant  $\kappa$  dependent only on  $B$ . Thus we have

$$\left| \int_t^{t+\varrho^2} \left( e^{(t+\varrho^2-\tau)B} \sigma v(\tau) \right)^{(j)} d\tau \right| \leq \kappa \int_t^{t+\varrho^2} (t + \varrho^2 - \tau)^j |v(\tau)| d\tau \leq$$

(by Hölder's inequality)

$$\leq \kappa \|v\|_{L^2([t, T])} \varrho^{2j+1},$$

and this proves (3.4).  $\square$

Let us consider the control problem (3.2) one more time. Among the paths  $\gamma$  satisfying (3.2), one is often interested in one minimizing the *total cost*

$$\|v\|_{L^2([t, T])}^2 = \int_t^T |v(s)|^2 ds.$$

Classical control theory provides the explicit expression of an optimal control and of its cost (see, for instance, [32], Theor. 9.55).

**Lemma 3.7.** *The optimal control for problem (3.2) is given by*

$$\bar{v}(s) = \left( e^{-(s-t)B} \sigma \right)^T \mathcal{C}^{-1}(-(T-t)) \left( e^{-(T-t)B} y - x \right), \quad s \in [t, T].$$

*The corresponding minimal cost will be denoted by*

$$V(t, x; T, y) := \|\bar{v}\|_{L^2([t, T])}^2$$

*and is equal to*

$$V(t, x; T, y) = \langle \mathcal{C}^{-1}(T-t)(y - e^{(T-t)B} x), y - e^{(T-t)B} x \rangle.$$

*Proof of Theorem 3.5.* In order to use the previous versions of the Harnack inequality, we first notice that by assumption, for every  $z \in ]T - \tau, T[ \times \mathbb{R}^d$ ,  $u$  is a continuous and non-negative weak solution of (1.1) in  $Q_{\sqrt{\tau}}^+(z)$ . Next we fix  $x, y \in \mathbb{R}^d$ ,  $t \in ]T - \tau, T[$  and consider the solution

$\gamma$  of the control problem (3.2) corresponding to the optimal control  $\bar{v}$  given in Lemma 3.7. Moreover, we set  $c_2 = \left(\frac{r}{\kappa}\right)^2$  where  $r$  and  $\kappa$  are the constants in Theorem 3.1 and Lemma 3.6 respectively.

Now, if  $T \leq t + \tau\beta$  and  $\|\bar{v}\|_{L^2([t,T])}^2 \leq c_2$ , then by Lemma 3.6 we have

$$(T, y) \in P_{1,r,\sqrt{\tau}}(t, x) \cap \left([t, t + \tau\beta] \times \mathbb{R}^d\right) \subseteq P_{\beta,r,\sqrt{\tau}}(t, x)$$

and therefore by Lemma 3.4 we get

$$u(T, y) \leq Cu(t, x)$$

where  $C$  is the constant in Theorem 3.1, which depends only on  $M$  and  $B$ .

Viceversa, setting  $t_0 = t$  and

$$t_{j+1} = (t_j + \tau\beta) \wedge \inf\{s \in [t_j, T] \mid \|\bar{v}\|_{L^2([t_j,s])}^2 \geq c_2\},$$

we have that  $t_j = T$  for  $j \geq \frac{1}{\beta} + \frac{\|\bar{v}\|_{L^2([t,T])}^2}{c_2}$  and

$$(t_{j+1}, \gamma(t_{j+1})) \in P_{1,r,\sqrt{\tau}}(t_j, \gamma(t_j)) \cap \left([t_j, t_j + \tau\beta] \times \mathbb{R}^d\right) \subseteq P_{\beta,r,\sqrt{\tau}}(t_j, \gamma(t_j))$$

if  $t_j < T$ . By Lemma 3.4 we have

$$u(t_j, \gamma(t_j)) \leq Cu(t_{j-1}, \gamma(t_{j-1})),$$

which yields

$$u(T, y) \leq C^{\frac{1}{\beta} + \frac{1}{c_2} V(t,x;T,y)} u(t, x).$$

The thesis follows by using the expression of the optimal cost given in Lemma 3.7.  $\square$

## 4 Lower bounds for fundamental solutions

The proof of the lower bound for the fundamental solution will make use of the following upper bound obtained in [33] and [25].

**Theorem 4.1 (Gaussian upper bound).** *Let  $\mathcal{L} \in \mathcal{K}_{M,B}$ . There exists a positive constant  $c_3$ , only dependent on  $M$  and  $B$ , such that*

$$\Gamma(t, x; T, y) \leq \frac{c_3}{(T-t)^{\frac{Q}{2}}} \exp\left(-\frac{1}{c_3} \left|\mathcal{D}\left((T-t)^{-\frac{1}{2}}\right) \left(y - e^{(T-t)B}x\right)\right|^2\right), \quad (4.1)$$

for  $0 < T-t \leq 1$  and  $x, y \in \mathbb{R}^d$ .

**Lemma 4.2.** *Let  $\mathcal{L} \in \mathcal{K}_{M,B}$ . There exist two positive constants  $R$  and  $c_4$ , which depend only on  $M$  and  $B$ , such that*

$$\int_{|\mathcal{D}(\sqrt{T-t})(y - e^{(T-t)B}x)| \leq R} \Gamma(t, x; T, y) dx \geq c_4, \quad 0 < T-t \leq 1, \quad y \in \mathbb{R}^d. \quad (4.2)$$

*Proof.* First notice that, for a suitably large constant  $c_5$  dependent only on  $M$  and  $B$ , the function

$$v(T, y) := \int_{\mathbb{R}^d} \Gamma(t, x; T, y) dx - e^{-c_5(T-t)}, \quad T > t, \ y \in \mathbb{R}^d,$$

is a weak super-solution of the forward Cauchy problem

$$\begin{cases} \mathcal{L}^* v(T, y) = -e^{-c_5(T-t)} (c - \text{tr}(B) + c_5) \leq 0, & T > t, \ y \in \mathbb{R}^d, \\ v(t, y) = 0 & y \in \mathbb{R}^d, \end{cases}$$

for the adjoint operator  $\mathcal{L}^*$  in (2.9). Therefore, by the maximum principle (see, for instance, Proposition 3.4 in [12]), we have  $v \geq 0$  that is

$$\int_{\mathbb{R}^d} \Gamma(t, x; T, y) dx \geq e^{-c_5(T-t)}, \quad T > t, \ y \in \mathbb{R}^d.$$

Then (4.2) follows from the following estimate:

$$\int_{|\mathcal{D}(\sqrt{T-t})(y - e^{(T-t)B}x)| \geq R} \Gamma(t, x; T, y) dx \leq$$

(by the upper bound (4.1))

$$\leq \frac{c_3}{(T-t)^{\frac{Q}{2}}} \int_{|\mathcal{D}(\sqrt{T-t})(y - e^{(T-t)B}x)| \geq R} \exp\left(-\frac{1}{c_3} \left| \mathcal{D}\left((T-t)^{-\frac{1}{2}}\right) (y - e^{(T-t)B}x) \right|^2\right) dx =$$

(by the change of variable  $z = \mathcal{D}\left((T-t)^{-\frac{1}{2}}\right) (y - e^{(T-t)B}x)$ )

$$= c_3 \int_{|z| \geq R} \exp\left(-\frac{1}{c_3} |z|^2\right) dz$$

which gives the thesis.  $\square$

We are now ready to state and prove the main result of the present paper.

**Theorem 4.3 (Gaussian lower bound).** *Let  $\mathcal{L} \in \mathcal{K}_{M,B}$ . There exists a positive constant  $C$ , dependent only on  $M$  and  $B$ , such that*

$$\Gamma(t, x; T, y) \geq \frac{C}{(T-t)^{\frac{Q}{2}}} e^{-\frac{1}{C} \langle C^{-1}(T-t)(y - e^{(T-t)B}x), y - e^{(T-t)B}x \rangle}, \quad 0 < T-t \leq 1, \ x, y \in \mathbb{R}^d. \quad (4.3)$$

**Remark 4.4.** *In general, estimate (4.3) is valid for any  $T-t > 0$ , with  $C$  dependent also on  $1 \vee (T-t)$ .*

*Proof.* We prove a preliminary diagonal estimate. Let  $\tau = \frac{T-t}{2}$ : by the global Harnack inequality stated in Theorem 3.5, for any  $\xi, y \in \mathbb{R}^d$  we have

$$\Gamma(t, y; T, y) \geq c_0 e^{-c_0 \langle C^{-1}(\tau)(\xi - e^{\tau B}y), \xi - e^{\tau B}y \rangle} \Gamma(t + \tau, \xi; T, y). \quad (4.4)$$

For any  $y \in \mathbb{R}^d$  we set

$$D_R = \{\xi \in \mathbb{R}^d \mid |\mathcal{D}(\sqrt{\tau})(y - e^{\tau B}\xi)| \leq R\}, \quad R > 0,$$

and notice that, up to a constant dependent only on  $M$  and  $B$ , the Lebesgue measure of  $D_R$  equals  $\tau^Q$ . We also note that, by Lemma 3.3 in [26],  $\langle \mathcal{C}^{-1}(\tau)(\xi - e^{\tau B}y), \xi - e^{\tau B}y \rangle$  is bounded on  $D_R$ . Therefore, integrating (4.4) over  $D_R$ , we get

$$\Gamma(t, y; T, y) \geq \frac{c_8}{\tau^Q} \int_{|\mathcal{D}(\sqrt{\tau})(y - e^{\tau B}\xi)| \leq R} \Gamma(t + \tau, \xi; T, y) d\xi \geq \frac{c_9}{(T - t)^{\frac{Q}{2}}}, \quad (4.5)$$

where the last inequality follows from Lemma 4.2 and the constant  $c_9$  depends only on  $M$  and  $B$ . Hence, by applying again the global Harnack inequality we get

$$\Gamma(t, 0; T, y) \geq c_0 e^{-c_0 \langle \mathcal{C}^{-1}(\tau)y, y \rangle} \Gamma(t + \tau, y; T, y) \geq$$

(by (4.5))

$$\geq \frac{c_{10}}{(T - t)^{\frac{Q}{2}}} e^{-c_0 \langle \mathcal{C}^{-1}(\tau)y, y \rangle} \geq \frac{c_{11}}{(T - t)^{\frac{Q}{2}}} e^{-c_{11} \langle \mathcal{C}^{-1}(T-t)y, y \rangle},$$

where the last inequality is a consequence of (4.7) from Remark 4.5 below. This proves (4.3) for  $x = 0$ ; the general statement follows by the translation-invariance property of the operator  $\mathcal{L}$ .  $\square$

**Remark 4.5.** *If we denote by  $\mathcal{C}_0$  the covariance matrix appearing in the fundamental solution of the homogenous principal part of  $\mathcal{L}$ , then there exist  $\alpha_1, \dots, \alpha_4, \beta_1, \dots, \beta_4 > 0$  such that for any  $\tau \in ]0, 1]$  and  $z \in \mathbb{R}^d$*

$$\alpha_1 \tau^Q \leq \alpha_2 \det(\mathcal{C}_0(\tau)) \leq \det(\mathcal{C}(\tau)) \leq \alpha_3 \det(\mathcal{C}_0(\tau)) \leq \alpha_4 \tau^Q \quad (4.6)$$

and

$$\beta_1 \left| \mathcal{D} \left( \tau^{-\frac{1}{2}} \right) z \right|^2 \leq \beta_2 \langle \mathcal{C}_0^{-1}(\tau) z, z \rangle \leq \langle \mathcal{C}^{-1}(\tau) z, z \rangle \leq \beta_3 \langle \mathcal{C}_0^{-1}(\tau) z, z \rangle \leq \beta_4 \left| \mathcal{D} \left( \tau^{-\frac{1}{2}} \right) z \right|^2. \quad (4.7)$$

In fact, we recall (see Proposition 2.3 in [26]) that for any  $\tau > 0$  one has

$$\mathcal{C}_0(\tau) = \mathcal{D}(\sqrt{\tau}) \mathcal{C}_0(1) \mathcal{D}(\sqrt{\tau})$$

and

$$\mathcal{C}_0^{-1}(\tau) = \mathcal{D} \left( \tau^{-\frac{1}{2}} \right) \mathcal{C}_0^{-1}(1) \mathcal{D} \left( \tau^{-\frac{1}{2}} \right).$$

These identities imply that

$$\det \mathcal{C}_0(\tau) = \det \left( \mathcal{D}(\sqrt{\tau}) \mathcal{C}_0(1) \mathcal{D}(\sqrt{\tau}) \right) = \tau^Q \det \mathcal{C}_0(1);$$

moreover, if  $k_1$  and  $k_2$  denote, respectively, the least and the greatest eigenvalue of  $\mathcal{C}_0^{-1}(1)$ , we have that  $k_1 > 0$  and

$$k_1 \left| \mathcal{D} \left( \tau^{-\frac{1}{2}} \right) z \right|^2 \leq \langle \mathcal{C}_0^{-1}(\tau) z, z \rangle \leq k_2 \left| \mathcal{D} \left( \tau^{-\frac{1}{2}} \right) z \right|^2$$

for all  $z \in \mathbb{R}^d$  and  $\tau > 0$ . This proves the first and last inequalities in (4.6) and (4.7). To prove the equivalence between the matrices  $\mathcal{C}_0^{-1}$  and  $\mathcal{C}^{-1}$ , we recall that, according to formula (3.14) in [26], we have

$$\frac{\det \mathcal{C}(\tau)}{\det \mathcal{C}_0(\tau)} = 1 + \tau O(1), \quad \text{as } \tau \rightarrow 0^+.$$

Hence  $\frac{\det \mathcal{C}(\tau)}{\det \mathcal{C}_0(\tau)}$  can be extended to a positive and continuous function on  $[0, 1]$ : consequently there exist two positive constants  $k_3$  and  $k_4$  such that

$$k_3 \det \mathcal{C}_0(\tau) \leq \det \mathcal{C}(\tau) \leq k_4 \det \mathcal{C}_0(\tau), \quad \tau \in [0, 1].$$

By the same argument we can prove that there exist two positive constants  $k_5$  and  $k_6$  such that

$$k_5 \langle \mathcal{C}_0^{-1}(\tau)z, z \rangle \leq \langle \mathcal{C}^{-1}(\tau)z, z \rangle \leq k_6 \langle \mathcal{C}_0^{-1}(\tau)z, z \rangle$$

for every  $z \in \mathbb{R}^d$  and  $\tau \in [0, 1]$  (see inequality (2.12) in [13]). Indeed we recall that, for every  $z \in \mathbb{R}^d$ ,

$$\langle \mathcal{C}^{-1}(\tau)z, z \rangle = \langle \mathcal{C}^{-1}(\tau)z, z \rangle = 1 + \tau O(1), \quad \text{as } \tau \rightarrow 0^+.$$

(see Lemma 3.3 in [26].) Then, the function  $(\tau, z) \mapsto \frac{\langle \mathcal{C}^{-1}(\tau)z, z \rangle}{\langle \mathcal{C}_0^{-1}(\tau)z, z \rangle}$  can be extended to a positive and continuous function on the compact set

$$\left\{ (\tau, z) \in \mathbb{R}^{d+1} \mid 0 \leq \tau \leq 1, |z| = 1 \right\}.$$

Then we conclude as above.

The following corollary is a straightforward consequence of Theorem 4.1 and Remark 4.5

**Corollary 4.6.** *Let  $\mathcal{L} \in \mathcal{K}_{M,B}$ . There exists a positive constant  $C$ , only dependent on  $M$  and  $B$ , such that*

$$\Gamma(t, x; T, y) \leq \frac{C}{\sqrt{\det \mathcal{C}(T-t)}} \exp \left( -\frac{1}{C} \langle \mathcal{C}^{-1}(T-t) \left( y - e^{(T-t)B} x \right), \left( y - e^{(T-t)B} x \right) \rangle \right).$$

for  $0 < T-t \leq 1$  and  $x, y \in \mathbb{R}^d$ .

## References

- [1] F. ANTONELLI, E. BARUCCI, AND M. E. MANCINO, *Asset pricing with a forward-backward stochastic differential utility*, Econom. Lett., 72 (2001), pp. 151–157.
- [2] F. ANTONELLI AND A. PASCUCCHI, *On the viscosity solutions of a stochastic differential utility problem*, J. Differential Equations, 186 (2002), pp. 69–87.
- [3] D. G. ARONSON, *Bounds for the fundamental solution of a parabolic equation*, Bull. Amer. Math. Soc., 73 (1967), pp. 890–896.

- [4] E. BARUCCI, S. POLIDORO, AND V. VESPRI, *Some results on partial differential equations and Asian options*, Math. Models Methods Appl. Sci., 11 (2001), pp. 475–497.
- [5] M. BOSSY, J.-F. JABIR, AND D. TALAY, *On conditional McKean Lagrangian stochastic models*, Probab. Theory Related Fields, 151 (2011), pp. 319–351.
- [6] C. CERCIGNANI, *The Boltzmann equation and its applications*, Springer-Verlag, New York, 1988.
- [7] C. CINTI, A. PASCUCCI, AND S. POLIDORO, *Pointwise estimates for a class of non-homogeneous Kolmogorov equations*, Math. Ann., 340 (2008), pp. 237–264.
- [8] E. B. DAVIES, *Explicit constants for Gaussian upper bounds on heat kernels*, Amer. J. Math., 109 (1987), pp. 319–333.
- [9] F. DELARUE AND S. MENOZZI, *Density estimates for a random noise propagating through a chain of differential equations*, J. Funct. Anal., 259 (2010), pp. 1577–1630.
- [10] L. DESVILLETES AND C. VILLANI, *On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: the linear Fokker-Planck equation*, Comm. Pure Appl. Math., 54 (2001), pp. 1–42.
- [11] M. DI FRANCESCO AND A. PASCUCCI, *On a class of degenerate parabolic equations of Kolmogorov type*, AMRX Appl. Math. Res. Express, 3 (2005), pp. 77–116.
- [12] M. DI FRANCESCO, A. PASCUCCI, AND S. POLIDORO, *The obstacle problem for a class of hypoelliptic ultraparabolic equations*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 464 (2008), pp. 155–176.
- [13] M. DI FRANCESCO AND S. POLIDORO, *Schauder estimates, Harnack inequality and Gaussian lower bound for Kolmogorov-type operators in non-divergence form*, Adv. Differential Equations, 11 (2006), pp. 1261–1320.
- [14] M. ELEUTERI AND S. POLIDORO, *Harnack inequality*, preprint, (2017).
- [15] E. B. FABES, *Gaussian upper bounds on fundamental solutions of parabolic equations; the method of Nash*, in Dirichlet forms (Varenna, 1992), vol. 1563 of Lecture Notes in Math., Springer, Berlin, 1993, pp. 1–20.
- [16] N. GAROFALO AND E. LANCONELLI, *Level sets of the fundamental solution and Harnack inequality for degenerate equations of Kolmogorov type*, Trans. Amer. Math. Soc., 321 (1990), pp. 775–792.
- [17] F. GOLSE, C. IMBERT, C. MOUHOT, AND A. VASSEUR, *Harnack inequality for kinetic Fokker-Planck equations with rough coefficients and application to the Landau equation*, preprint, arXiv.org:1607.08068, (2016).

- [18] F. GOLSE AND A. VASSEUR, *Hölder regularity for hypoelliptic kinetic equations with rough diffusion coefficients*, preprint, Version 2, (2015).
- [19] D. G. HOBSON AND L. C. G. ROGERS, *Complete models with stochastic volatility*, Math. Finance, 8 (1998), pp. 27–48.
- [20] L. HÖRMANDER, *Hypoelliptic second order differential equations*, Acta Math., 119 (1967), pp. 147–171.
- [21] A. M. IL'IN, *On a class of ultraparabolic equations*, Dokl. Akad. Nauk SSSR, 159 (1964), pp. 1214–1217.
- [22] C. IMBERT AND C. MOUHOT, *Hölder continuity of solutions to hypoelliptic equations with bounded measurable coefficients*, preprint, arXiv.org:1505.04608, Version 5, (2015).
- [23] L. P. KUPCOV, *Mean value theorem and a maximum principle for Kolmogorov's equation*, Mathematical notes of the Academy of Sciences of the USSR, 15 (1974), pp. 280–286.
- [24] ———, *On parabolic means*, Dokl. Akad. Nauk SSSR, 252 (1980), pp. 296–301.
- [25] A. LANCONELLI AND A. PASCUCCI, *Nash estimates and upper bounds for non-homogeneous Kolmogorov equations*, to appear in Potential Anal. DOI 10.1007/s11118-017-9622-1, (2017).
- [26] E. LANCONELLI AND S. POLIDORO, *On a class of hypoelliptic evolution operators*, Rend. Sem. Mat. Univ. Politec. Torino, 52 (1994), pp. 29–63.
- [27] P. LANGEVIN, *On the theory of Brownian motion*, C. R. Acad. Sci. (Paris), 146 (1908), pp. 530–533.
- [28] P.-L. LIONS, *On Boltzmann and Landau equations*, Philos. Trans. Roy. Soc. London Ser. A, 346 (1994), pp. 191–204.
- [29] J. MOSER, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math., 17 (1964), pp. 101–134.
- [30] ———, *Correction to: “A Harnack inequality for parabolic differential equations”*, Comm. Pure Appl. Math., 20 (1967), pp. 231–236.
- [31] J. NASH, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math., 80 (1958), pp. 931–954.
- [32] A. PASCUCCI, *PDE and martingale methods in option pricing*, vol. 2 of Bocconi & Springer Series, Springer, Milan; Bocconi University Press, Milan, 2011.
- [33] A. PASCUCCI AND S. POLIDORO, *A Gaussian upper bound for the fundamental solutions of a class of ultraparabolic equations*, J. Math. Anal. Appl., 282 (2003), pp. 396–409.



- [34] ———, *The Moser's iterative method for a class of ultraparabolic equations*, Commun. Contemp. Math., 6 (2004), pp. 395–417.
- [35] R. PESZEK, *PDE models for pricing stocks and options with memory feedback*, Appl. Math. Finance, 2 (1995), pp. 211–223.
- [36] S. POLIDORO, *On a class of ultraparabolic operators of Kolmogorov-Fokker-Planck type*, Matematiche (Catania), 49 (1994), pp. 53–105.
- [37] ———, *A global lower bound for the fundamental solution of Kolmogorov-Fokker-Planck equations*, Arch. Rational Mech. Anal., 137 (1997), pp. 321–340.
- [38] H. RISKEN, *The Fokker-Planck equation: Methods of solution and applications*, Springer-Verlag, Berlin, second ed., 1989.
- [39] I. M. SONIN, *A class of degenerate diffusion processes*, Teor. Veroyatnost. i Primenen., 12 (1967), pp. 540–547.
- [40] W. WANG AND L. ZHANG, *The  $C^\alpha$  regularity of weak solutions of ultraparabolic equations*, Discrete Contin. Dyn. Syst., 29 (2011), pp. 1261–1275.
- [41] M. WEBER, *The fundamental solution of a degenerate partial differential equation of parabolic type*, Trans. Amer. Math. Soc., 71 (1951), pp. 24–37.